Consider the linear non-homogeneous second order differential equation

$$y'' + p(x) \cdot y' + q(x) \cdot y = g(x). \tag{1}$$

We assume that p, q, g are continuous on some interval I so that we can guarantee the existence of solutions. The general solution to this equation is of the form $y(x) = y_h(x) + y_p(x)$ where $y_h(x)$ is the general solution to the associated homogeneous equation and $y_p(x)$ is a particular solution. If $p(x)$ and $q(x)$ are constant functions, then we've seen how the method of undetermined coefficients can be effective in obtaining $y_p(x)$. On the other hand, there are some obvious limitations to this method (e.g., how well can you guess at the form of the particular solution, etc).

When the method of undetermined coefficients seems too unwieldily to apply, we can turn to another method. Much like the method of reduction of order, we start by assuming that the particular solution y_p is some *variable* combination of a fundamental set of solutions to the associated homogeneous equation, say y_1, y_2 (which must be known ahead of time):

$$y_p(x) = c_1(x) \cdot y_1(x) + c_2(x) \cdot y_2(x).$$

After plugging this expression into 1 and simplifying, we arrive at a pair of equations

$$\begin{aligned} 0 &= c_1'(x)y_1(x) + c_2'(x)y_2(x) \\ g(x) &= c_1'(x)y_1'(x) + c_2'(x)y_2'(x) \end{aligned}$$

In matrix form, we have

$$\begin{bmatrix} 0 \\ g(x) \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \cdot \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix}.$$

Using the inverse of the matrix above we obtain

$$\begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} =$$

Upon integrating, we come to the following important idea:

Variation of Parameters

Let y_1 and y_2 be a fundamental set of solutions to the associated homogeneous equation for 1. Then the particular solution $y_p(x)$ to 1 is given by

$$-y_1(x) \int_{x_0}^{x} \frac{y_2(t)g(t)}{W(t)}\, \mathrm{d}t + y_2(x) \int_{x_0}^{x} \frac{y_1(t)g(t)}{W(t)}\, \mathrm{d}t,$$

where W is the Wronskian of y_1, y_2 and x_0 is a conveniently chosen constant (typically, x_0 is chosen to match the initial condition if one is present).

Example: Use the method of variation of parameters to find the general solution to $y'' - 2y' - 3y = \cos(2t)$.

HW: Due Wednesday, 6-Jul

1. Find the general solution to $y'' + 4y = 3\csc(2x)$ on the interval $0 < x < \pi/2$.

2. Consider the differential equation $x^2y'' - x(x+2)y' + (x+2)y = 2x^3$ with $x > 0$. Use the fact that $y_1(x) = x$ and $y_2(x) = xe^x$ form a fundamental set of solutions to the associated homogeneous equation to find a particular solution to the ODE under consideration.